Unsteady Heat Conduction

- In this section we will consider the process of unsteady heat conduction - the process by which a body reaches thermal equilibrium with its surroundings.
- The governing equation in this case (without heat generation) is:

$$\frac{dT}{dt} = \alpha \nabla^2 T \qquad \alpha = \frac{k}{\rho\, c_p} \equiv \text{thermal diffusivity}$$

- At steady state, the RHS is equal to zero.
- When perturbed by a change is boundary conditions, the RHS is no longer zero and the rate at which the temperature responds depends upon the magnitude of the thermal diffusivity, α .

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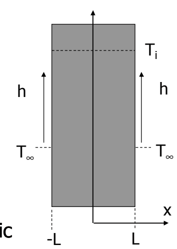
Unsteady Heat Conduction (cont)

- Consider the following case of a wall initial at equilibrium at T_0, suddenly exposed to a cooling convective stream at T_∞.
- Let's non-dimensionalize this problem by using the following definitions:

$$\theta(x,t) = \frac{T(x,t) - T_\infty}{T_0 - T_\infty}$$

$$\bar{x} = \frac{x}{L_c} \qquad \bar{t} = \frac{t}{t_c}$$



- Where L_c is a characteristic length (probably L), and t_c is a characteristic time period.

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Unsteady Heat Conduction (cont)

- Using these definitions, the governing equation may be re-written as:

$$\frac{d\theta}{d\bar{t}} = \frac{\alpha\, t_c}{L_c^2}\frac{d^2\theta}{d\bar{x}^2} = Fo\,\frac{d^2\theta}{d\bar{x}^2}$$

- The group of terms, ($\alpha t_c/L_c^2$), that is itself non-dimensional and is called the Fourier number, Fo.
- The Fourier number is a useful indicator of how a body has responded:
  - A low value of Fo indicates that the body has not responded to changes yet except possibly near the boundaries.
  - A high value of Fo indicates that the body has had large changes in T and may in fact be near reaching a new equilibrium.

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Unsteady Heat Conduction (cont)

- The boundary condition (x = ±L) for this case is:

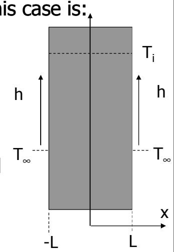
$$-k\frac{dT}{dx} = h(T - T_\infty)$$

- Or after non-dimensionalizing:

$$-\frac{d\theta}{d\bar{x}} = \frac{hL_c}{k}\theta$$

- This introduces a new non-dimensional term, hL_c/k, called the Biot number, Bi.
- Another way to write Bi is:

$$Bi = \frac{hL_c}{k} = \frac{(L_c/kA)}{(1/hA)} = \frac{R_{t,cond}}{R_{t,conv}}$$



Unsteady Heat Conduction (cont)

- Thus, Bi is the ratio of conduction to convection resistance. As a result:
  - A low value of Bi corresponds to a body with little internal temperature variation, but a large temperature jump at the surface (like copper).
  - A high value of Bi corresponds to a body with surface temperatures near the fluid temperature, but large internal temperature variations.
- As a result of this non-dimensional analysis we can write that:

$$\theta = f\left(\bar{x}, \bar{t}, Fo, Bi\right)$$

- In the solutions that follow, we will see this functionality appear repeatedly.

Unsteady Heat Conduction (cont)

- A couple of final notes about our new factors of Fo and Bi.
- For cases other than this simple planar wall, we define the characteristic length as the volume of the body divided by the convective surface area, A_s:

$$L_c = \frac{V}{A_s} \qquad \begin{aligned} L_c &= L \quad \text{(half width) for planar walls} \\ L_c &= R/2 \quad \text{for rods} \\ L_c &= R/3 \quad \text{for spheres} \end{aligned}$$

- The Fourier number is usually written in terms of the time itself rather than a characteristic time. Thus:

$$\boxed{Fo = \frac{\alpha t}{L_c^2}}$$

Lumped Heat Capacity

- The first case we will consider is where Bi << 1.
- In this case, the internal temperature variations are negligible and the body can be considered at a single temperature, T(t).
- In this case, the energy entering the body through convection increases it's energy (and T) by:

$$-hA_s\left(T-T_\infty\right)=\frac{dE}{dt}=c_p\rho V\frac{dT}{dt}$$

- Defining $\theta=\left(T-T_\infty\right)$, gives the governing equation:

$$\frac{c_p\rho V}{hA_s}\frac{d\theta}{dt}=-\theta \quad \text{with initial condition :} \quad \theta_0=\theta_{t=0}=\left(T_0-T_\infty\right)$$

Lumped Heat Capacity (cont)

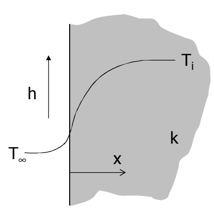
- This equation and boundary condition are satisfied by:

$$\frac{\theta}{\theta_0}=\frac{T-T_\infty}{T_0-T_\infty}=\exp\left[-\left(\frac{hA_s}{c_p\rho V}\right)t\right]$$

- Use this equation when ever heating a highly conductive material, a very small (thin) body, or where the convective cooling rate is very, very small.
- Note: The Fourier number doesn't appear in this eqn. Since the body is always in quasi-equilibrium internally. The rate process occurs on the surface, not the interior!

Semi-Infinite Solids

- The next approximate case to consider is the situation of a semi-infinite solid, as shown:
- This case is also that of a finite width solid at very small values of Fo (time) such that only the region near the convection boundary has had time to respond.
- Thus, the interior temperature is still at the initial value, T_i.



Semi-Infinite Solids (cont)

- The governing eqn. in this situation is:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

- However, a simplified equation can be obtained by assuming similarity. I.e. that solutions for all t and x may be represented by functions of a single parameter given by: $\eta = \frac{x}{2\sqrt{\alpha t}}$
- Since: $\frac{\partial \eta}{\partial t} = \frac{-x}{4t\sqrt{\alpha t}}$ $\left(\frac{\partial \eta}{\partial x}\right)^2 = \frac{1}{4\alpha t}$

  the governing eqn can be rewritten as:

Semi-Infinite Solids (cont)

- The governing eqn. in this situation is:

$$\frac{dT}{d\eta} = -2\eta \frac{d^2 T}{d\eta^2}$$

- This ODE occurs fairly regularly and has known solutions.
- For the most general boundary condition at x=0 where: $k\left.\frac{dT}{dx}\right|_{x=0} = h\left(T_{x=0} - T_\infty\right)$
- the solution is:

$$\frac{T(x,t)-T_i}{T_\infty - T_i} = erfc\left(\frac{x}{2\sqrt{\alpha t}}\right) - \exp\left(\frac{hx}{k} + \frac{h^2 \alpha t}{k^2}\right) erfc\left(\frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)$$

Semi-Infinite Solids (cont)

- Where the complimentary error function, erfc, is:

$$erfc(x) = 1 - erf(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} du$$

- Tabulations of the error function are given in Appendix A. However, the results are also plotted parametrically in Figure 4-5, pg 140.
- The curves in this figure are represented as:

$$\frac{T(x,t)-T_i}{T_\infty - T_i} = f\left(\frac{x}{2\sqrt{\alpha t}}, \frac{h\sqrt{\alpha t}}{k}\right)$$

- Do you see the Bi and Fo numbers buried in this functionality?

Semi-Infinite Solids (cont)

- A simpler solution exists for the case where convection is very high and you can assume the surface temperature is equal to the fluid temp.
- For this case, the initial condition and boundary condition at x=0 is:

$$T_i = T_{t=0} \qquad T_o = T_{x=0} = T_\infty$$

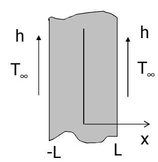
- the solution is:

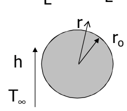
$$\frac{T(x,t) - T_0}{T_i - T_0} = erf\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

- A plot of these solutions if given on page 137 as Figure 4-4.

Finite Thickness Solids

- Finally, consider the more generalized case of a solid with finite dimensions and cooled equally on all sides.
- The book gives analytic solutions for these cases when T(x,0) = T_i.
- However, these solutions have also been presented graphically in Appendix C for plane walls, cylinders, and spheres.





Finite Thickness Solids (cont)

- The graphs in section 4-4 (Fig. 4-7 to 4-16) actually come in three types: the time variation of the center temperature, the spatial variation within the body, and the total heat lost (gained).
- The first plot, center temperature is in the form:

$$\frac{\theta_o}{\theta_i} = \frac{T(0,t) - T_\infty}{T_i - T_\infty} = f(Fo, Bi)$$

- As Fo $\rightarrow$0, this function goes to 1.0 indicating no change in the center temperature.
- Thus, for Fo < 0.2, these charts are inaccurate and the semi-infinite solid method should be used.

Finite Thickness Solids (cont)

- The spatial variation is shown in a chart of:

$$\frac{\theta}{\theta_o} = \frac{T(x,t) - T_\infty}{T(0,t) - T_\infty} = f(x/L, Bi)$$

- From these we see that the spatial variation is very small for Bi < .1. For these cases, the lumped heat capacity method would work just as well.
- The heat transfer charts show the variation in the current heat lost versus the final heat lost:

$$\frac{Q}{Q_o} = \frac{\int q dt}{mc(T_i - T_\infty)} = f(Fo, Bi)$$

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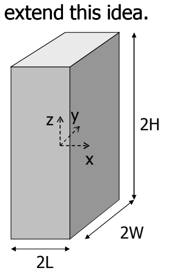
Multi-Dimensional Effects

- The previous method for slabs only accounted for a finite width in one dimension. Let's extend this idea.
- If we consider a generalized 3-D box, the governing eqn is:

$$\frac{\partial T}{\partial t} = \alpha \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right]$$



- Assume that separation of variables (in space) applies such that the solution will be:

$$\frac{\theta(x,y,z,t)}{\theta_i} = \frac{\theta_1(x,t)}{\theta_i} \cdot \frac{\theta_2(y,t)}{\theta_i} \cdot \frac{\theta_3(z,t)}{\theta_i}$$

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Multi-Dimensional Effects (cont)

- The solution for each component is that for a finite slab in that direction, I.e:

$$\frac{\theta_1(x,t)}{\theta_i} = f(x/L, Fo_L, Bi_L) \qquad Fo_L = \frac{\alpha t}{L^2} \qquad Bi_L = \frac{hL}{k}$$

$$\frac{\theta_2(y,t)}{\theta_i} = f(y/W, Fo_W, Bi_W) \qquad Fo_W = \frac{\alpha t}{W^2} \qquad Bi_W = \frac{hW}{k}$$

$$\frac{\theta_3(z,t)}{\theta_i} = f(z/H, Fo_H, Bi_H) \qquad Fo_H = \frac{\alpha t}{H^2} \qquad Bi_H = \frac{hH}{k}$$

- The solution for each spatial axis is found from the Heisler charts on pages 142-150, and the combined 3-D solution from the previous equation.

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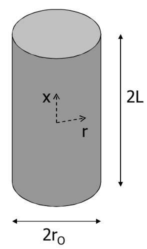
Multi-Dimensional Effects (cont)

- Another similar problem is that of a cylinder of finite length.
- This problem is 2-dimensional in r and x, such that:

$$\frac{\partial T}{\partial t} = \alpha \left[\frac{\partial^2 T}{\partial x^2} + \frac{1}{r}\frac{d}{dr}\left(\frac{1}{r}\frac{dT}{dr} \right) \right]$$

- And, with separation of the spatial variables, the solution will be in the form:

$$\frac{\theta(x,r,t)}{\theta_i} = \frac{\theta_1(x,t)}{\theta_i} \cdot \frac{\theta_2(r,t)}{\theta_i}$$



Multi-Dimensional Effects (cont)

- The solution for each component is that for a finite slab in that direction, I.e:

$$\frac{\theta_1(x,t)}{\theta_i} = f\left(x/L, Fo_L, Bi_L\right) \qquad Fo_L = \frac{\alpha\, t}{L^2} \qquad Bi_L = \frac{hL}{k}$$

$$\frac{\theta_2(r,t)}{\theta_i} = f\left(r/r_O, Fo_{r_O}, Bi_{r_O}\right) \qquad Fo_{r_O} = \frac{\alpha\, t}{r_O^{\,2}} \qquad Bi_{r_O} = \frac{hr_O}{k}$$

- As before, the solution in both spatial directions (x and r) are found and combined to obtained the 2-D solution.
- Can you see how to also get 2-D solutions for something like a very long rectangular rod?