

### Falkner-Skan Solutions

- The Blasius and Van Driest flat plate solutions show the power of using flow similarity in solving viscous flows.
- Unfortunately, there are only a very few situations for which similarity applies – most flows are too complex.
- There is, however, another class of similar flows which help us visualize the effect of pressure gradients – the Falkner-Skan family of flows.
- We will look at these flows in order to better grasp the impact of pressure gradients on laminar flows.
- Always keep in mind, however, that most flows are too complex for simple methods and usually require computational techniques for solutions.

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### Falkner-Skan Solutions (2)

- First, lets start with the B.L. equations including the pressure term we dropped for Blasius solution.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dp_e}{dx} + \nu \frac{\partial^2 u}{\partial y^2}$$

- The pressure return we will eliminate by using Euler’s equation in the freestream:

$$dp_e = -\rho V_\infty dV_\infty \Rightarrow -\frac{1}{\rho} \frac{dp_e}{dx} = V_\infty \frac{dV_\infty}{dx}$$

- Based upon what we learned from the Blasius solution we will assume that the horizontal velocity and y coordinate can be represented by:

$$u = V_\infty f'(\eta) \quad y = \frac{\eta}{g(x)}$$

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### Falkner-Skan Solutions (3)

- In the Blasius solution we used the stream function to automatically satisfy flow continuity.
- This time take a different approach and use continuity to eliminate the vertical velocity from our equation.

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \quad v(y) = \int_0^y \frac{\partial v}{\partial y} dy = -\int_0^y \frac{\partial u}{\partial x} dy = -\frac{\partial}{\partial x} \int_0^y u dy$$

- Were the boundary conditions have been used to evaluate the integral at y=0:

$$y=0 \Rightarrow u=v=0$$

$$y \rightarrow \infty \Rightarrow u \rightarrow u_e = V_\infty$$

- The remaining momentum equation is now:

$$u \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \frac{\partial}{\partial x} \int_0^y u dy = V_\infty \frac{dV_\infty}{dx} + \nu \frac{\partial^2 u}{\partial y^2}$$

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### Falkner-Skan Solutions (4)

- In expanding this equation, we need to expand the derivative in terms of the transformed variables:

$$\xi = x \quad \eta = yg(x)$$

- Realize that that the freestream velocity is no longer a constant but varies with  $x$  (or  $\xi$ ) and that we do not yet know the form of  $g$ :

$$\frac{\partial u}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial u}{\partial \eta} = V_\infty f' + yV_\infty g' f''$$

$$\frac{\partial u}{\partial y} = \frac{\partial \xi}{\partial y} \frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial u}{\partial \eta} = V_\infty g f'' \quad \frac{\partial^2 u}{\partial y^2} = V_\infty g^2 f'''$$

$$\int_0^y u dy = \int_0^y \frac{V_\infty}{g} f' d\eta = \frac{V_\infty}{g} f \quad \frac{\partial}{\partial x} \int_0^y u dy = \left( \frac{V_\infty'}{g} - \frac{V_\infty}{g^2} g' \right) f + \frac{yV_\infty g'}{g} f'$$

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### Falkner-Skan Solution (5)

- Putting this into the momentum equation gives:

$$V_\infty f''(V_\infty f' + yV_\infty g' f'') - V_\infty g f''' \left[ \left( \frac{V_\infty'}{g} - \frac{V_\infty}{g^2} g' \right) f + \frac{yV_\infty g'}{g} f' \right] =$$

$$V_\infty V_\infty' + yV_\infty g^2 f'''$$

- Or, after rearranging:

$$f''' = \frac{V_\infty g'}{V_\infty g^3} f f'' + (f'^2 - f f'' - 1) \frac{V_\infty'}{V_\infty g^2}$$

- In order to have similarity, all dependence upon  $x$  must disappear from the above expression – or rather, the two multipliers above must be equal to constants:

$$\frac{V_\infty g'}{V_\infty g^3} = \text{constant} \quad \frac{V_\infty'}{V_\infty g^2} = \text{constant}$$

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### Falkner-Skan Solution (6)

- Falner and Skan found that this could be achieved if the transform and velocity were expressed by power laws:

$$g = Cx^a \quad V_\infty = Kx^m \quad m = 2a + 1$$

- The also choose the constants  $C$  and  $K$  to be compatible with Blasius for the case of zero pressure gradient,  $m=0$ :

$$f''' + \frac{f f''}{2} = 0$$

- In this case:

$$C^2 = K \frac{1+m}{\nu} \quad \eta = y \sqrt{(1+m) \frac{V_\infty}{\nu x}}$$

- And the governing equation becomes:

$$f''' + \frac{f f''}{2} + \frac{\beta}{2} (1 - f'^2) = 0 \quad \beta = \frac{2m}{1+m}$$

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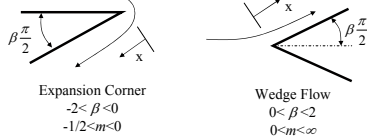
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## Falkner-Skan Wedge Flows

- It is natural to ask what exactly are the flows described by the power law type velocity variation:

$$V_\infty = Kx^m$$

- It turns out that this equation describes the flow over wedges.



- Note the first case is a decelerating velocity, i.e. adverse pressure gradient, while the second case is accelerating and thus favorable.

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## Falkner-Skan Wedge Flows (2)

- The wedge flow case has some usefulness.
- The expansion corner, by contrast, is not realistic since the B.L. does not begin until AFTER the corner.
- However, both cases provide tremendous insight into the behavior of laminar B.L.'s under a pressure gradient.
- The solutions for Falkner-Skan flow for  $\beta$  from near 2 to -0.1988 are shown on the following page.
- The limit of  $\beta \rightarrow 2$  ( $m \rightarrow \infty$ ) is for an extremely rapidly accelerating flow.
- Also, while the shape has a similar shape, the B.L. thickness is decreasing as velocity increases.

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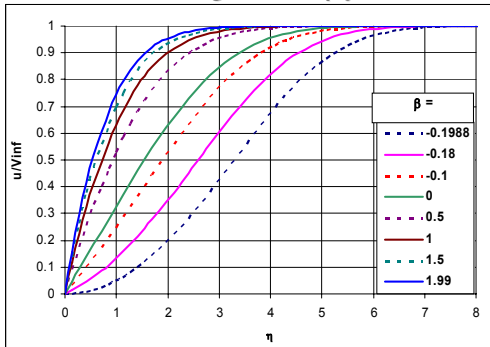
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## Falkner-Skan Wedge Flows (3)




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### Falkner-Skan Wedge Flows (4)

- The other limit,  $\beta \rightarrow -0.1988$  ( $m \rightarrow -0.0904$ ) is the point of incipient separation.
- At this point, the velocity gradient at the wall is zero – i.e. there is no wall shear stress.
- Beyond this point there is no possible solution to the equation – which tells us that separated flows are not similar in nature.
- This makes sense since separated flows must have a separation point – and thus cannot have the same shape before and after separation.

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### Momentum Integral Equation

- Before leaving laminar flow and moving on to turbulence, there is one other special equation of note.
- Begin with the incompressible momentum equation with the integral continuity equation replacing the vertical velocity:

$$u \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \int_0^y u dy = V_\infty \frac{dV_\infty}{dx} + \nu \frac{\partial^2 u}{\partial y^2}$$

- No consider if we integrated this momentum equation across the B.L. – from the wall to the freestream:

$$\int_0^\infty \left( u \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} \int_0^y u dy - V_\infty \frac{dV_\infty}{dx} \right) dy = \int_0^\infty \nu \frac{\partial^2 u}{\partial y^2} dy$$

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### Momentum Integral Equation (2)

- The right hand side term integrates directly to give:

$$\int_0^\infty \nu \frac{\partial^2 u}{\partial y^2} dy = \frac{\mu}{\rho} \frac{\partial u}{\partial y} \Big|_0^\infty = -\frac{\tau_w}{\rho}$$

- Also, the middle term on the left hand side can be integrated by parts:

$$\int_0^\infty \left( \frac{\partial u}{\partial y} \int_0^y u dy \right) dy = V_\infty \int_0^\infty \frac{\partial u}{\partial x} dy - \int_0^\infty u \frac{\partial u}{\partial x} dy$$

- With these two expressions, our integral equation becomes:

$$\int_0^\infty \left( 2u \frac{\partial u}{\partial x} - V_\infty \frac{\partial u}{\partial x} - V_\infty \frac{dV_\infty}{dx} \right) dy = -\frac{\tau_w}{\rho}$$

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### Momentum Integral Equation (3)

- The new middle term can be expanded:

$$V_\infty \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(V_\infty u) - u \frac{dV_\infty}{dx}$$

- So that the integral equation can be rewritten as:

$$\int_0^\infty \left[ \frac{\partial}{\partial x}(u^2 - V_\infty u) + \frac{dV_\infty}{dx}(u - V_\infty) \right] dy = -\frac{\tau_w}{\rho}$$

- Now compare this with the definitions of displacement and momentum thickness:

$$\delta^* = \int_0^\infty \left( 1 - \frac{u}{V_\infty} \right) dy \quad \theta = \int_0^\infty \frac{u}{V_\infty} \left( 1 - \frac{u}{V_\infty} \right) dy$$

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### Momentum Integral Equation (4)

- From the comparison, we see that our final equation can be rewritten in terms of  $\delta^*$  and  $\theta$  as:

$$\frac{d}{dx}(V_\infty^2 \theta) + \delta^* \frac{dV_\infty}{dx} = \frac{\tau_w}{\rho}$$

- This is the **Momentum Integral Equation**, an simplified form of the incompressible Boundary Layer Equations.
- Because we integrated across the B.L., this equation does not involve the details of the B.L. shape.
- In fact, solutions to this equation can be thought of as satisfy the original B.L. equations on average rather than exactly.

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### Pohlhausen Solution

- To solve the Momentum Integral Equation, we need to relate all the variables,  $V_\infty$ ,  $\delta^*$ ,  $\theta$  and  $\tau_w$ , by assuming a shape to the B.L.
- A popular approach was proposed by Pohlhausen who used a quadratic equation for his family of B.L.'s in the form:

$$\frac{u}{V_\infty} = (2\eta - 2\eta^3 + \eta^4) + \frac{\Lambda}{6}(\eta - 3\eta^2 + 3\eta^3 - \eta^4) \quad \eta = \frac{y}{\delta}$$

- Which satisfies the boundary conditions:

$$y = 0 \quad \Rightarrow \quad u = 0 \quad \frac{\partial^2 u}{\partial y^2} = V_\infty \frac{\partial V_\infty}{\partial x}$$

$$y \rightarrow \infty \quad \Rightarrow \quad u = V_\infty \quad \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = 0$$

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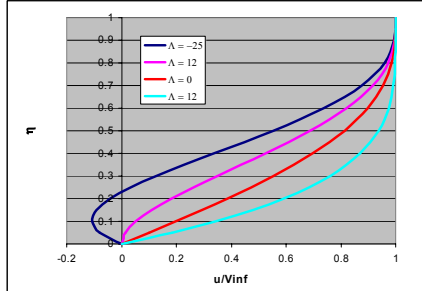
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### Pohlhausen Solution (2)

- The factor  $\Lambda$ , determines the B.L. profile shape and depends upon the velocity gradient in the freestream:

$$\Lambda = \frac{\rho \delta^2}{\mu} \frac{dV_\infty}{dx}$$




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### Pohlhausen Solution (3)

- The combination of the Momentum Integral Equation with the assumed shape results in an ODE in  $x$ .
- This equation can be numerically marched in the  $x$  direction starting with suitable initial conditions.
- The results, while approximate, yields pretty accurate solutions for the B.L. thicknesses and shear stress – for considerably less effort than solving the original PDE's.
- For a flat plate, the results are:

$$\frac{\delta^*}{x} = \frac{1.752}{\sqrt{Re_x}} \quad \frac{\theta}{x} = \frac{0.686}{\sqrt{Re_x}} \quad c_f = \frac{1.372}{\sqrt{Re_x}}$$

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### Pohlhausen Solution (4)

- Another important observation about Pohlhausen's solution, without solving it, is about separation.
- The family of B.L. shapes shows that separation occurs for  $\Lambda \sim -12$ .

But:

$$\Lambda = \frac{\rho \delta^2}{\mu} \frac{dV_\infty}{dx}$$

- Thus, we see that we must have a negative velocity gradient (positive pressure gradient) for separation.
- As important, we see that "young" B.L.'s, where  $\delta$  is small, can withstand higher gradients before separating.
- "Old" B.L.'s which are thicker, will separate earlier.

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