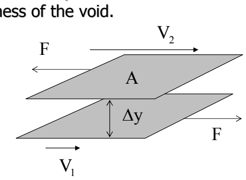
Intro. To Fluid Viscosity

- The basic concept in fluid viscosity is **Newton's Law**:
  - When a fluid fills the void between two surfaces with different velocities, a force exists on each surface that is directly proportional to the surface area and velocity difference but inversely proportional to the thickness of the void.
  - Mathematically:



$$\boxed{F \propto A\Delta V / \Delta y}$$

$F \equiv$ Force (N or lbs)

$A \equiv$ area (m^2 or ft^2)

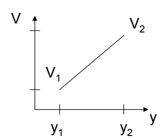
$\Delta V \equiv$ velocity difference $= V_2 - V_1$ (ft/sec)

$\Delta y \equiv$ thickness (m or ft)

Intro. To Fluid Viscosity [2]

- The constant of proportionality is called the viscosity, μ (kg/m/sec or slug/ft/sec), so that:

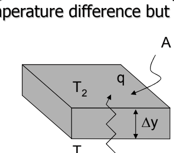
$$\boxed{F = \mu A \frac{\Delta V}{\Delta y} = \mu A \frac{dV}{dy}}$$



  - Note that experimentation shows that there would be a linear variation of velocity in this situation!
- Usually, we will be interested in the shear stress – the shear force per unit area: $\boxed{\tau = \frac{F}{A} = \mu \frac{dV}{dy}}$

Intro. To Conduction

- The basic concept in heat conduction is **Fourier's Law**:
  - When two differing temperatures occur on opposing sides of a material, the rate of heat transfer through the material is directly proportional to the surface area and temperature difference but inversely proportional to the thickness.
  - Mathematically:



$$\boxed{q \propto A\Delta T / \Delta y}$$

$q \equiv$ heat transfer rate or heat flux (W or ft - lb/sec)

$A \equiv$ area (m^2 or ft^2)

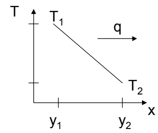
$\Delta T \equiv$ temperature difference $= T_2 - T_1$ (^{o}K or ^{o}R)

$\Delta y \equiv$ thickness (m or ft)

Intro. To Conduction [2]

- The constant of proportionality is called the thermal conductivity, k (W/m/°K or lb/sec/°R), so that:

$$q = -kA\frac{\Delta T}{\Delta y} = -kA\frac{dT}{dy}$$



  - Where the negative sign is necessary if q is positive when flowing in the positive y direction, but $dT/dy < 0$!
- Usually, we will be interested in the heat flux – the heat transfer per unit area:

$$\dot{q} = \frac{q}{A} = -k\frac{dT}{dy}$$

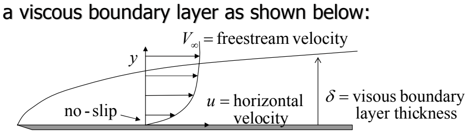
Laminar Versus Turbulent

- The previous relations are valid for either laminar or turbulent flow.
- However, for laminar flow, both viscosity and conductivity occur at a molecular level due to the random motion of the particles.
- In turbulent flow, the chaotic mixing of the flow, much more efficient than particle motions, greatly increases both the viscosity and conductivity.
- As a result, we could separate the coefficients into laminar and turbulent components – the former being a property of the gas, the later a property of the flow:

$$\tau = (\mu + \mu_t)\frac{dV}{dy} \qquad \dot{q} = -(k + k_t)\frac{dT}{dy}$$

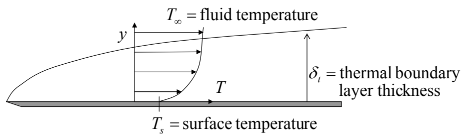
Introduction to Laminar Boundary Layers

- Finally, lets look at the topology of the interaction between the wall and the fluid.
- We will initially restrict ourselves to laminar boundary layers and assume steady flow conditions.
- We will also just consider the case of flat plate flow to avoid complications due to surface curvature.
- The viscous no-slip condition at the wall interface results in a viscous boundary layer as shown below:



Introduction to Laminar BL's [2]

- At the same time, a thermal boundary layer develops as heat is transferred between the wall and fluid:



- Note that at the surface, T(x,y,z) = T_s - the thermal equivalent to no-slip.
- This thermal equilibrium is due to both the need for continuity and the low (zero!) flow velocities near the wall allowing time to equilibrate.

Introduction to Laminar BL's[3]

- The goal of viscous analysis will be to determine either the wall shear stress and/or the wall heat transfer.
- These values at any point will be given by:

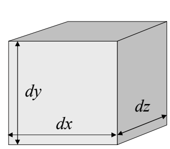
$$\tau_{xy} = \mu\left(\frac{\partial V}{\partial y}\right)_{\text{wall}} \qquad \dot{q} = -k\left(\frac{\partial T}{\partial y}\right)_{\text{wall}}$$

  - The subscripts on the shear stress are meant to indicate a shear in the x direction due to a velocity gradient in the y direction.
- These values will have to be integrated over the wall to determine the overall force (drag) and/or the heat transfer between wall and flow.

Conservation Equations

- Since we now know that we need to solve the fluid problem, let's look a the governing equations.
- We will need to conserve mass, momentum (3-directions), and energy in the boundary layer (B.L.).
- If we consider a small control volume in the B.L., a general statement of conservation for steady state is:

$$\begin{bmatrix}\text{The net rate at which}\\ \text{a property flow out}\end{bmatrix} = \begin{bmatrix}\text{The rate of production}\\ \text{of that property inside}\end{bmatrix}$$



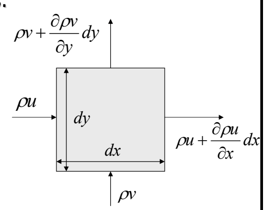
Conservation Equations [2]

- Some textbooks go through a length development of conservations equations for the most general case: 3-D, compressible, fully viscous.
- The resulting very complex, equations are usually known as the Navier-Stokes Equations.
- In class, we will do a simpler development by making assumptions valid for most boundary layers.
- I.e.
  - 2-D flow
  - Viscous action normal to the surface only
- The resulting equations are called the Boundary Layer Equations and only apply to the viscous layer near the surface!

Mass Conservation

- Apply this conservation rule to mass for the 2-D case below.
- The net rate of mass outflow is:

$$\left(\rho u + \frac{\partial \rho u}{\partial x}dx\right)dy + \left(\rho v + \frac{\partial \rho v}{\partial y}dy\right)dx - \rho\, u dy - \rho\, v dy$$



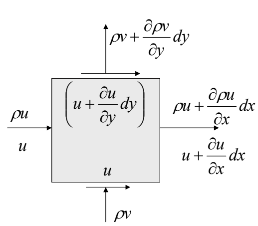
- Since there is no mass production, we get:

$$\boxed{\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0}$$

Momentum Conservation

- Now consider momentum conservation - in particular conservation of X momentum.
- In the diagram shown, realize that ρu is the property, while both u and v are flux rates!
- Thus, the next flux out is:

$$\left(u + \frac{\partial u}{\partial x}dx\right)\left(\rho u + \frac{\partial \rho u}{\partial x}dx\right)dy + \left(u + \frac{\partial u}{\partial y}dy\right)\left(\rho v + \frac{\partial \rho v}{\partial y}dy\right)dx - \rho\, u^2 dy - \rho\, uv dx$$



Momentum Conservation [2]

- After expanding, canceling terms, and dropping products of differentials (like $\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}$) as negligible, we get:

$$\text{X momentum flux} = \left(u\frac{\partial \rho u}{\partial x} + \rho u\frac{\partial u}{\partial x} + u\frac{\partial \rho v}{\partial y} + \rho v\frac{\partial u}{\partial y} \right) dxdy$$

- Or, since by mass conservation:

$$\frac{\partial \rho u}{\partial x} = -\frac{\partial \rho v}{\partial y}$$

- Doing the same process for Y momentum gives:

$$\text{X momentum flux} = \left(\rho\, u\frac{\partial u}{\partial x} + \rho\, v\frac{\partial u}{\partial y} \right) dxdy$$

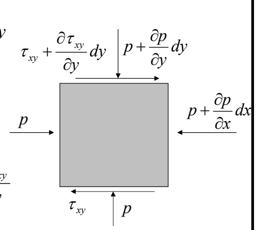
$$\text{Y momentum flux} = \left(\rho\, v\frac{\partial v}{\partial y} + \rho\, u\frac{\partial v}{\partial x} \right) dxdy$$

Momentum Conservation [3]

- Momentum is produced by forces acting on the C.V.
- If we consider only pressure forces and shear in the x direction, then:

$$\text{X production} = \left(-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right) dxdy$$

$$\text{Y production} = -\frac{\partial p}{\partial y} dxdy$$



- Setting net flux equal to production gives:

$$\text{X}: \quad \rho\, u\frac{\partial u}{\partial x} + \rho\, v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}$$

$$\text{Y}: \quad \rho\, v\frac{\partial v}{\partial y} + \rho\, u\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y}$$

Momentum Conservation [4]

- We can now apply Newton's law of viscosity:

$$\tau_{xy} = \mu\frac{\partial u}{\partial y} \qquad \mu = \text{laminar viscosity}$$

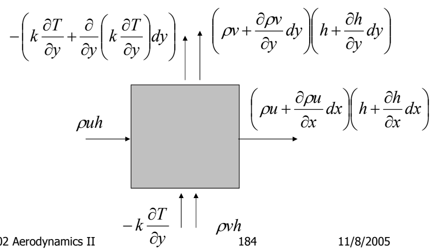
- Also, experimental evidence shows that in most B.L.'s, both dp/dy and v are very small. As a result, the Y momentum eqn. is much less important than the X.
- Thus, it is sufficient in a boundary layer to express momentum conservation by:

$$\rho\, u\frac{\partial u}{\partial x} + \rho\, v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y} \right)$$

$$\frac{\partial p}{\partial y} = 0$$

Energy Conservation

- Lastly, consider the fluxes of energy out of our C.V.
- Note that there will not only be advection (energy carried by flow motion), but also conduction away from the surface!



Energy Conservation [2]

- Summing the fluxes gives:

$$\left(\rho u+\frac{\partial \rho u}{\partial x}dx\right)\left(h+\frac{\partial h}{\partial x}dx\right)dy+\left(\rho v+\frac{\partial \rho v}{\partial y}dy\right)\left(h+\frac{\partial h}{\partial y}dy\right)dx$$
$$-\left(k\frac{\partial T}{\partial y}+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)dy\right)dx-\rho uhdy-\rho vhdx+-k\frac{\partial T}{\partial y}dx$$

- Or after canceling and dropping higher order terms:

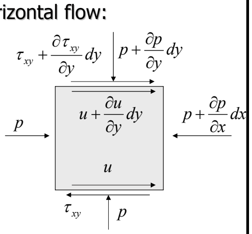
$$\left[\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}+h\left(\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}\right)^{\nearrow 0}-\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\right]dxdy$$

- But, by mass conservation, the 3rd term is zero!

Energy Conservation [3]

- Energy is generated by the rate at which work is done by surface forces – pressure and friction.
- Since we are assuming negligible vertical velocities, there is only work due to the horizontal flow:

$$\left(u+\frac{\partial u}{\partial y}dy\right)\left(\tau_{xy}+\frac{\partial \tau_{xy}}{\partial y}dy\right)dx$$
$$+\left(u+\frac{\partial u}{\partial x}dx\right)\left(p+\frac{\partial p}{\partial x}dx\right)dx$$
$$-u\tau_{xy}dx-updx$$



- Or, after canceling terms:

$$\mu\left(\frac{\partial u}{\partial y}\right)^2 dxdy+u\frac{\partial p}{\partial x}$$

Boundary Layer Equations

- Thus, our energy conservation is just:

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}-\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)=u\frac{\partial p}{\partial x}+\mu\left(\frac{\partial u}{\partial x}\right)^2$$

- And to summarize all of our conservation equations:

$$\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}=0$$

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{dp_e}{dx}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$$

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}=\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+u\frac{dp_e}{dx}+\mu\left(\frac{\partial u}{\partial y}\right)^2$$

Boundary Layer Equations

  - Note that the pressure gradient has been written to emphasize that pressure is only a function of x and is equal to the pressure at the edge of the boundary layer, p_e.

Navier-Stokes Equations

- By comparison, the 2-D Navier-Stokes equations look like:

$$\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}=0$$

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial x}\left[\lambda\left(\nabla\cdot\vec{V}\right)+2\mu\frac{\partial u}{\partial x}\right]$$

$$\rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[\lambda\left(\nabla\cdot\vec{V}\right)+2\mu\frac{\partial v}{\partial y}\right]$$

$$\rho u\frac{\partial h}{\partial x}+\rho v\frac{\partial h}{\partial y}=\nabla\cdot\left(k\nabla T\right)+\nabla\cdot\left(p\vec{V}\right)+\frac{\partial}{\partial x}\left[v\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[u\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]$$

$$+\frac{\partial}{\partial x}\left[u\left(\lambda\left(\nabla\cdot\vec{V}\right)+2\mu\frac{\partial u}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[v\left(\lambda\left(\nabla\cdot\vec{V}\right)+2\mu\frac{\partial v}{\partial y}\right)\right]$$

Non-Dimensional Equations

- Before attempt to solve the governing Boundary Layer equations just derived, consider what can be learned by non-dimensionalizing them.
- To obtain non-dimension distances velocities, divide by some characteristic length and the freestream velocities, respectively:

$$\bar{x}=\frac{x}{L}\qquad \bar{y}=\frac{y}{L}\qquad \bar{u}=\frac{u}{V_\infty}\qquad \bar{v}=\frac{v}{V_\infty}$$

- For temperature , density and pressure, also use freestream conditions:

$$\bar{\rho}=\frac{\rho}{\rho_\infty}\qquad \bar{T}=\frac{T}{T_\infty}\qquad \bar{p}=\frac{p_e}{\gamma p_\infty}$$

- Also assume thermally perfect and isotropic properties:

$$dh=c_p dT\qquad \mu=\text{constant}\qquad k=\text{constant}$$

Non Dimensional Equations [2]

- While the other properties are constants and are left alone (including density!).
- After inserting these definitions into our B.L. eqns:

$$\frac{\partial \bar{\rho}\bar{u}}{\partial \bar{x}} + \frac{\partial \bar{\rho}\bar{v}}{\partial \bar{y}} = 0$$

$$\bar{\rho}\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{\rho}\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\left(\frac{\gamma p_\infty}{\rho_\infty V_\infty^2}\right)\frac{d\bar{p}}{d\bar{x}} + \left(\frac{\mu}{\rho_\infty V_\infty L}\right)\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

$$\bar{\rho}\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{\rho}\bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \left(\frac{k}{\rho_\infty c_p V_\infty L}\right)\frac{\partial^2 \bar{T}}{\partial \bar{y}^2} + \left(\frac{\gamma p_\infty}{\rho_\infty c_p T_\infty}\right)\bar{u}\frac{d\bar{p}}{d\bar{x}} + \left(\frac{\mu V_\infty}{\rho_\infty c_p L T_\infty}\right)\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2$$

- Now consider the grouping of terms inside the brackets.

Non Dimensional Equations [3]

- From gas dynamics, the first term resolves into a function of the Mach number, M:

$$\frac{\gamma p_\infty}{\rho_\infty V_\infty^2} = \frac{a_\infty^2}{V_\infty^2} = \frac{1}{M^2} \qquad M = \frac{V_\infty}{a_\infty}$$

- Literally, the Mach number is the ratio of the flow velocity to the speed of sound, a.
- Practically, the Mach number is a good measure of the compressibility of a fluid.
- For M < 0.3, the fluids density is constant for all practical purposes.
- For M > 0.3, changes in density must be taken into account.

Non Dimensional Equations [4]

- The second grouping of terms is called the Reynolds number, Re, based on the length L:

$$\frac{\mu}{\rho_\infty V_\infty L} = \frac{\upsilon}{V_\infty L} = \frac{1}{\mathrm{Re}_L} \qquad \mathrm{Re}_L = \frac{\rho_\infty V_\infty L}{\mu}$$

- The Reynolds number is the ratio of flow momentum to fluid viscosity.
- For low Re's (<100), viscosity dominates the flow and the viscous boundary layer is very thick.
- At high Re's (> 1e6), the B.L. becomes thin and viscous effects are only important near the surface.
- Note also the term $\upsilon=\mu/\rho$ which is called either the kinematic viscosity or the viscous diffusivity.

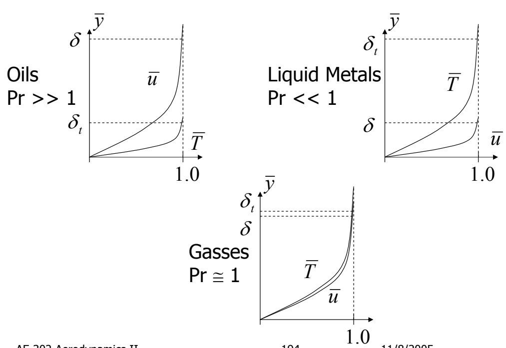
Non Dimensional Equations [5]

- The next group of terms can be written as the product of Reynolds number and a new term, the Prandlt number, Pr.

$$\frac{k}{\rho_\infty c_p V_\infty L} = \frac{\mu}{\rho_\infty V_\infty L}\left(\frac{k}{c_p \mu}\right) = \frac{1}{\mathrm{Re}_L}\left(\frac{1}{\mathrm{Pr}}\right) \qquad \boxed{\mathrm{Pr} = \frac{c_p \mu}{k}}$$

- The Prandlt number is the ratio of viscous to thermal diffusivities.
- When large (as in oils), the viscous boundary layer is much larger than the thermal boundary layer.
- When small (as in liquid metals), the opposite is true.
- When Pr ~ 1 (as in gasses), they are about equal in size!

Non Dimensional Equations [6]



Non Dimensional Equations [7]

- Note that the product of Re_L and Pr may be used by itself and is called the Peclet number, $\boxed{\mathrm{Pe} = \mathrm{Re}_L \, \mathrm{Pr}}$
- And, the last term is composed of the Reynolds number and the Eckert number, Ec:

$$\frac{\mu V_\infty}{\rho_\infty c_p L T_\infty} = \frac{\mu}{\rho_\infty V_\infty L}\left(\frac{V_\infty^2}{c_p T_\infty}\right) = \frac{\mathrm{Ec}}{\mathrm{Re}_L} \qquad \boxed{\mathrm{Ec} = \frac{V_\infty^2}{c_p\left(T_s - T_\infty\right)}}$$

- The Eckert number is the ratio of the flow energy to the B.L. enthalpy difference.
- Generally, the Eckert number is useful in cases where heating from flow friction (aeroheating) becomes important.

Non Dimensional Equations [8]

- With these definitions, our B.L. equations become:

$$\frac{\partial \bar{\rho}\bar{u}}{\partial \bar{x}} + \frac{\partial \bar{\rho}\bar{v}}{\partial \bar{y}} = 0$$

$$\bar{\rho}\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{\rho}\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = -\frac{1}{M^2}\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{1}{\mathrm{Re}_L}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

$$\bar{\rho}\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{\rho}\bar{v}\frac{\partial \bar{T}}{\partial \bar{y}} = \frac{1}{\mathrm{Re}_L \mathrm{Pr}}\frac{\partial^2 \bar{T}}{\partial \bar{y}^2} + (\gamma - 1)\bar{u}\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\mathrm{Ec}}{\mathrm{Re}_L}\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2$$

- Thus, when solved, we would get solutions of u, v and T in the form:

$$\bar{u} = f(\mathrm{M}, \mathrm{Re}_L) \qquad \bar{v} = f(\mathrm{M}, \mathrm{Re}_L)$$

$$\bar{T} = f(\mathrm{M}, \mathrm{Re}_L, \mathrm{Pr}, \mathrm{Ec}, \gamma)$$