## Range and Endurance

- When designing or comparing aircraft, two parameters usually come to mind:
  - **Range**: the horizontal distance an airplane can travel on a single fueling. The cruise portion of a flight is associated with flying for range.
  - **Endurance**: the amount of time an airplane can remain aloft on a single fueling. The loiter phase of a mission is associated with flying for endurance.
- We will calculate the range and endurance for piston-propeller and turbojet aircraft separately.
- All our calculations will assume still air conditions - I.e. no head or tail winds.



## R and E - Propeller Aircraft

- Both range and endurance of an aircraft will depend upon the rate at which fuel is burned.
- The common parameter used to define this rate is called the **Specific Fuel Consumption**, SFC.

$$SFC \equiv \frac{\text{Weight of fuel}}{\text{Engine Power} \cdot \text{Time}} = \frac{\text{lb of fuel}}{\text{(bhp) hour}}$$

- The SFC is considered a constant for an engine type - varying vary little with throttle setting or flight conditions.
- Typical range of values: 0.4 - 0.7 lb/hp/hr



## R and E - Propeller Aircraft (continued)

- From the fact that the SFC is a constant, we can deduce some simple range and endurance relations.
- For a long endurance, we would like to burn the minimum fuel per hour. From the SFC then:

$$\frac{\text{lb of fuel}}{\text{hour}} \propto SFC \cdot (bhp) \propto \frac{SFC \cdot P_R}{\eta}$$

- Thus, for long endurance we want to fly with a high propeller efficiency and at minimum **power** required!
- From our previous result for power required, it follows that for max endurance, we want to fly at the velocity such that $C_L^{3/2}/C_D$ is maximum.

## R and E - Propeller Aircraft (continued)

- For a long range, we would like to burn the minimum fuel per mile traveled. Dividing our previous result by velocity (miles/hour) gives:

$$\frac{\text{lb of fuel}}{\text{mile}} \propto \frac{SFC \cdot (bhp)}{V_\infty} \propto \frac{SFC \cdot T_R}{\eta}$$

- Thus, for long range we want to fly with a high propeller efficiency and at minimum **thrust** required!
- From our previous result for thrust required, it follows that for max range, we want to fly at the velocity such that $C_L/C_D$ is maximum.

(Remember, these results are for propeller driven aircraft - jets will be different!)



## R and E - Propeller Aircraft (continued)

- Now let's work at some quantitative relations.
- First, since SFC does not have consistent units, a new fuel burn rate is introduced:

$$c \equiv \frac{\text{lb of fuel}}{(\text{ft} \cdot \text{lb/sec})\,\text{sec}} = \frac{1}{\text{ft}} \qquad \left( c \equiv \frac{\text{N of fuel}}{(W)\,\text{sec}} = \frac{1}{\text{m}} \right)$$

- Note that in the British system, c and SFC are related via the conversion factors:

$$c = \frac{SFC}{\left(550\,\frac{ft \cdot lb}{\sec \cdot hp}\right)\left(3600\,\frac{\sec}{hour}\right)} = \frac{SFC}{\left(1980000\,\frac{ft \cdot lb}{hour \cdot hp}\right)}$$

- Is other references, if these conversion factors are part of the equations, they are using SFC not c!



## R and E - Propeller Aircraft (continued)

- The weight of an aircraft is usually broken down into various components of which only the fuel portion varies during normal flight.

$$W = W_{OEW} + W_{Payload} + W_f$$

- The rate at which fuel is burned depends upon the SFC and the power setting:

$$dW = dW_f = -cPdt$$

- Rearrange gives $dt = dW/-cP$ which can be integrated to get the endurance:

$$E = \int_{t_0}^{t_1} dt = \int_{W_0}^{W_1} \frac{dW}{-cP} \qquad \boxed{E = \int_{W_1}^{W_0} \frac{dW}{cP}}$$

## R and E - Propeller Aircraft (continued)

- Similarly, the range is found by multiplying time by velocity and integrating:

$$dx = V_\infty dt = \frac{V_\infty dW}{-cP}$$

$$R = \int_{x_0}^{x_1} dx = \int_{W_0}^{W_1} \frac{V_\infty dW}{-cP} \qquad \boxed{R = \int_{W_1}^{W_0} \frac{V_\infty dW}{cP}}$$

- These integral relations are useful for calculating the R and E for a given mission where V and P may vary through the flight.
- However, they are not very useful for rapid R and E estimations for a given airplane. For that purpose, we use the Breguet relations!



## Breguet Propeller Equations

- The Breguet relations are approximate expressions for R and E obtained by making assumptions about a typical flight profile.
- For range, assume steady, level flight so that L=W and $P_R=V_\infty T_R=V_\infty D$.

$$R = \int_{W_1}^{W_0} \frac{V_\infty dW}{cP} = \int_{W_1}^{W_0} \frac{\eta V_\infty W dW}{c V_\infty D W} = \int_{W_1}^{W_0} \frac{\eta L}{cD} \frac{dW}{W}$$

- Now assume that c and $\eta$ are constant, and that the aircraft is flown at a velocity such that $L/D = C_L/C_D$ remains a constant.
- (Note that $V_\infty$ and $\rho_\infty$ are not necessarily constant!)



## Breguet Propeller Eqns (continued)

- With these assumptions, the integral can be evaluated to get:

$$\boxed{R = \frac{\eta}{c} \frac{C_L}{C_D} \ln\left(\frac{W_o}{W_1}\right)}$$

- This relations tells us that to maximize range, we want:
  - The largest possible propeller efficiency
  - The lowest possible specific fuel consumption
  - The largest possible weight fraction of fuel, $W_f/W$
  - A large $(L/D)_{max}$ and to fly at the velocity where this is achieved.

## Breguet Propeller Eqns (continued)

- Perform a similar procedure with the endurance equation.

$$E = \int_{W_1}^{W_0} \frac{dW}{cP} = \int_{W_1}^{W_0} \frac{\eta W dW}{c V_\infty D W} = \int_{W_1}^{W_0} \frac{\eta L}{c V_\infty D} \frac{dW}{W}$$

- To go further, we need to relate velocity to lift via:

$$V_\infty = \sqrt{2W/(\rho_\infty S C_L)}$$

$$E = \int_{W_1}^{W_0} \frac{\eta C_L}{c C_D} \sqrt{\frac{\rho_\infty S C_L}{2W}} \frac{dW}{W} = \int_{W_1}^{W_0} \frac{\eta C_L^{3/2}}{c C_D} \sqrt{\frac{\rho_\infty S}{2}} \frac{dW}{W^{3/2}}$$

- Now assume that c, $\eta$ and $\rho_\infty$ are constant, and that the aircraft is flown at a velocity such that $C_L^{3/2}/C_D$ remains a constant



## Breguet Propeller Eqns (continued)

- With these assumption, integration yields:

$$\boxed{E = \frac{\eta C_L^{3/2}}{c C_D} \sqrt{2 \rho_\infty S} \left( \frac{1}{W_1^{1/2}} - \frac{1}{W_0^{1/2}} \right)}$$

- This relations tells us that to maximize endurance, we want:
  - The largest possible propeller efficiency
  - The lowest possible specific fuel consumption
  - The largest possible weight fraction of fuel, $W_f/W$
  - A large $(C_L^{3/2}/C_D)_{max}$ and to fly at the velocity where this is achieved.
  - Flying at the highest density possible, I.e. sea level!



## R and E - Jet Aircraft

- Jet aircraft differ from propeller aircraft primarily in the fact that a jet engine produces **thrust** directly while a piston engine produces **power**.
- The fuel consumption for jet aircraft is thus based upon T and called the **Thrust Specific Fuel Consumption**, TSFC.

$$TSFC \equiv \frac{\text{Weight of fuel}}{(\text{Engine Thrust})\text{hour}} = \frac{\text{lb of fuel}}{(\text{lb of thrust})\text{hour}}$$

- As with piston engines, TSFC is a nearly a constant for a given powerplant.
- Typical range of values: 0.5 - 1.0 lb/lb/hr

## R and E - Jet Aircraft (continued)

- From the fact that the TSFC is a constant, we can deduce some simple range and endurance relations.
- For a long endurance, we would like to burn the minimum fuel per hour. From the TSFC then:

$$\frac{\text{lb of fuel}}{\text{hour}} \propto TSFC \cdot (\text{lb of thrust})$$

- Thus, for long endurance on a jet, we want to fly with a minimum thrust required!
- From our previous result for thrust required, it follows that for max endurance, we want to fly at the velocity such that $C_L/C_D$ is maximum.



## R and E - Jet Aircraft (continued)

- For a long range, we would like to burn the minimum fuel per mile traveled. Dividing our previous result by velocity (miles/hour) gives:

$$\frac{\text{lb of fuel}}{\text{mile}} \propto \frac{TSFC \cdot (\text{lb of thrust})}{V_\infty} \propto \frac{TSFC \cdot T_R}{V_\infty}$$

- To reach a conclusion from this relation, lets assume $T_R$=D and substitute velocity from our $C_L$ definition

$$\frac{T_R}{V_\infty} = \frac{D}{V_\infty} = \tfrac{1}{2}\rho_\infty V_\infty S C_D = \tfrac{1}{2}\rho_\infty \sqrt{\frac{2W}{\rho_\infty S C_L}} S C_D = \sqrt{\tfrac{1}{2}\rho_\infty S W}\,\frac{C_D}{C_L^{1/2}}$$

- Thus, among other things, for max range we want to fly at a velocity such that $C_L^{1/2}/C_D$ is a maximum!



## R and E - Jet Aircraft (continued)

- Now let's work at the quantitative relations.
- First, since TSFC does not have consistent units, a new fuel burn rate is introduced:

$$c_t \equiv \frac{\text{lb of fuel}}{(\text{lb of thrust})\sec} = \frac{1}{\sec} \qquad \left( c_t \equiv \frac{\text{N of fuel}}{(N)\sec} = \frac{1}{\sec} \right)$$

- Note that $c_t$ is the same in either unit system. Converting to TSFC results in:

$$c_t = \frac{TSFC}{\left(3600\,\frac{\sec}{hour}\right)}$$

- As before, always check units and conversion factors when using other references!

## R and E - Jet Aircraft (continued)

- The rate at which jets burn fuel is given by:

$$dW = dW_f = -c_t T dt$$

- Rearranging and integrating for endurance gives:

$$E = \int_{t_0}^{t_1} dt = \int_{W_0}^{W_1} \frac{dW}{-c_t T} \qquad \boxed{E = \int_{W_1}^{W_0} \frac{dW}{c_t T}}$$

- Similarly, range will be calculated by:

$$R = \int_{x_0}^{x_1} dx = \int_{W_0}^{W_1} \frac{V_\infty dW}{-c_t T} \qquad \boxed{R = \int_{W_1}^{W_0} \frac{V_\infty dW}{c_t T}}$$

- Note how similar these equations are to those for propellers - only the denominator is different.



## Breguet Jet Equations

- Now make assumptions similar to those for propellers previously to get the Breguet Jet equations from these integral relations.
- For endurance, assume steady, level flight so that L=W and $T=T_R=D$.

$$E = \int_{W_1}^{W_0} \frac{dW}{c_t T} = \int_{W_1}^{W_0} \frac{W dW}{c_t D W} = \int_{W_1}^{W_0} \frac{L}{c_t D} \frac{dW}{W}$$

- Now with $c_t$ a constant and assuming the aircraft is flown at a velocity such that $L/D = C_L/C_D$ remains a constant.



## Breguet Jet Eqns (continued)

- Pulling out theses constants and integrating yields:

$$\boxed{E = \frac{1}{c_t} \frac{C_L}{C_D} \ln\left(\frac{W_o}{W_1}\right)}$$

- This relations tells us that to maximize endurance, we want:
  - The lowest possible thrust specific fuel consumption
  - The largest possible weight fraction of fuel, $W_f/W$
  - A large $(L/D)_{max}$ and to fly at the velocity where this is achieved.

## Breguet Jet Eqns (continued)

- Perform a similar procedure with the jet range equation.

$$R = \int_{W_1}^{W_0} \frac{V_\infty dW}{c_T T} = \int_{W_1}^{W_0} \frac{V_\infty W dW}{c_t D W} = \int_{W_1}^{W_0} \frac{V_\infty L}{c_t D} \frac{dW}{W}$$

- To go further, relate velocity to lift via:

$$V_\infty = \sqrt{2W/(\rho_\infty S C_L)}$$

$$R = \int_{W_1}^{W_0} \frac{C_L}{c_t C_D} \sqrt{\frac{2W}{\rho_\infty S C_L}} \frac{dW}{W} = \int_{W_1}^{W_0} \frac{C_L^{1/2}}{c_t C_D} \sqrt{\frac{2}{\rho_\infty S}} \frac{dW}{W^{1/2}}$$

- Now assume that $c_t$ and $\rho_\infty$ are constant, and that the aircraft is flown at a velocity such that $C_L^{1/2}/C_D$ remains a constant



## Breguet Jet Eqns (continued)

- With these assumptions, integration yields:

$$R = \frac{2}{c_t} \frac{C_L^{1/2}}{C_D} \sqrt{\frac{2}{\rho_\infty S}} \left(W_0^{1/2} - W_1^{1/2}\right)$$

- This relations tells us that to maximize range, we want:
  - The lowest possible specific fuel consumption
  - The largest possible weight fraction of fuel, $W_f/W$
  - A large $(C_L^{1/2}/C_D)_{max}$ and to fly at the velocity where this is achieved.
  - Flying at the lowest density possible, I.e. high altitude!



## Breguet Eqns - Summary

- Here is a summary of the Breguet equations:
- For piston-propellers:

$$R = \frac{\eta}{c} \frac{C_L}{C_D} \ln\left(\frac{W_o}{W_1}\right) \qquad E = \frac{\eta C_L^{3/2}}{c C_D} \sqrt{2\rho_\infty S} \left(\frac{1}{W_1^{1/2}} - \frac{1}{W_0^{1/2}}\right)$$

- For turbojets:

$$R = \frac{2}{c_t} \frac{C_L^{1/2}}{C_D} \sqrt{\frac{2}{\rho_\infty S}} \left(W_0^{1/2} - W_1^{1/2}\right) \qquad E = \frac{1}{c_t} \frac{C_L}{C_D} \ln\left(\frac{W_o}{W_1}\right)$$

## $C_L$ and $C_D$ relations - Summary

- For $(C_L^{3/2}/C_D)_{max}$, $C_{D,i}=3C_{D,0}$

$$C_L = \sqrt{3\pi e AR C_{D,0}} \qquad \frac{C_L^{3/2}}{C_D} = \frac{\left(3\pi e AR C_{D,0}\right)^{3/4}}{4C_{D,0}}$$

- For $(C_L/C_D)_{max}$, $C_{D,i}=C_{D,0}$

$$C_L = \sqrt{\pi e AR C_{D,0}} \qquad \frac{C_L}{C_D} = \frac{1}{2}\sqrt{\frac{\pi e AR}{C_{D,0}}}$$

- For $(C_L^{1/2}/C_D)_{max}$, $C_{D,i}=1/3\ C_{D,0}$

$$C_L = \sqrt{\tfrac{1}{3}\pi e AR C_{D,0}} \qquad \frac{C_L^{1/2}}{C_D} = \frac{\left(\tfrac{1}{3}\pi e AR C_{D,0}\right)^{1/4}}{\tfrac{4}{3}C_{D,0}}$$