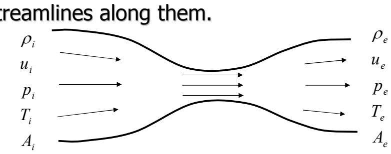
Variable Area Internal Flow

- Let's return now to internal flow, but now with area as a variable rather than constant.
- As shown below, this type flow is not 1-D anymore due to the curved wall surfaces and the streamlines along them.



- As problematic is the fact that the flow properties are not constant across the duct.

Variable Area Internal Flow [2]

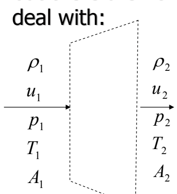
- However, a good approximation can be made by assuming uniform properties and horizontal flow – the Quasi 1-D Flow assumption.
- This is the same approximation used in fluid mechanics to obtain the pipe continuity equation:

$$\rho VA = \text{constant}$$

- In practice the longer the duct and thus the less curved the walls, the better the approximation.
- Short ducts with rapid area changes will not be well modeled – but those are usually not good flow situations anyway.

Quasi 1-D Flow Equations

- If a segment of the duct is used with known initial conditions, we can use our conservation equations to develop relations for the outflow conditions.
- The problem is very similar to the one for 1-D flow, but there are now two areas – both known – to deal with:



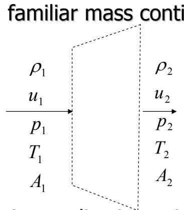
$$\oiint_S \rho \overline{V} \cdot \hat{n} dS = 0$$

$$\oiint_S \rho \overline{V}\left(\overline{V} \cdot \hat{n}\right) dS + \oiint_S p\hat{n} dS = 0$$

$$\oiint_S \rho\left(e + \tfrac{1}{2}V^2\right)\overline{V} \cdot \hat{n} dS + \oiint_S p\left(\overline{V} \cdot \hat{n}\right) dS = 0$$

Quasi 1-D Flow Equations [2]

- Applying the integral relations to the inflow and outflows (no flow through the sides) gives the familiar mass continuity equation:



$$\oiint_S \rho \overline{V} \cdot \hat{n} dS = 0$$

$$-\rho_1 u_1 A_1 + \rho_2 u_2 A_2 = 0$$

$$\boxed{\rho_1 u_1 A_1 = \rho_2 u_2 A_2}$$

- A complication exists in applying the momentum equation because while there is not flow through the sides, there is a variable pressure there.

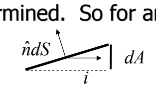
Quasi 1-D Flow Equations [3]

- Without yet knowing how pressure varies with area, that term must be left as an integral:

$$\oiint_S \rho \overline{V}\left(\overline{V} \cdot \hat{n}\right) dS + \oiint_S p\hat{n} dS = 0$$

$$-p_1 A_1 - \rho_1 u_1^2 A_1 + p_2 A_2 + \rho_2 u_2^2 A_2 + \oiint_{sides} p(i \cdot \hat{n}) dS = 0$$

- Note that the sign of the dot product can be determined. So for an increasing area:



$$\left(i \cdot \hat{n}\right) dS = -dA$$

- Thus:

$$\boxed{p_1 A_1 + \rho_1 u_1^2 A_1 + \iint_{A_1}^{A_2} p dA = p_2 A_2 + \rho_2 u_2^2 A_2}$$

Quasi 1-D Flow Equations [4]

- Finally, the energy conservation equation gives the same result as before:

$$\oiint_S \rho\left(e + \tfrac{1}{2}V^2\right)\overline{V} \cdot \hat{n} dS + \oiint_S p\left(\overline{V} \cdot \hat{n}\right) dS = 0$$

$$-\rho_1\left(e_1 + \tfrac{1}{2}u_1^2\right)u_1 A_1 - p_1 u_1 A_1 + \rho_2\left(e_2 + \tfrac{1}{2}u_2^2\right)u_2 A_2 + p_2 u_2 A_2 = 0$$

$$-\rho_1 u_1 A_1\left(e_1 + \frac{p_1}{\rho_1} + \tfrac{1}{2}u_1^2\right) + \rho_2 u_2 A_2\left(e_2 + \frac{p_2}{\rho_2} + \tfrac{1}{2}u_2^2\right) = 0$$

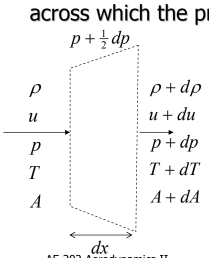
$$e_1 + \frac{p_1}{\rho_1} + \tfrac{1}{2}u_1^2 = e_2 + \frac{p_2}{\rho_2} + \tfrac{1}{2}u_2^2$$

$$\boxed{h_1 + \tfrac{1}{2}u_1^2 = h_2 + \tfrac{1}{2}u_2^2}$$

- Note that by our Quasi 1-D assumption, there is not normal velocity component – thus $u = V$!

Differential Equation Forms

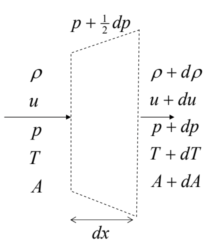
- Because of the remaining integral term, the previous set of equation cannot be solved yet.
- Instead, choose a new, very thin control volume across which the properties change infinitesimally:



- We will also assume that the pressures on the sides can be represented adequately by the average of the initial and final pressures.
- This assumption is consistent with a linear approximation of the changes.

Differential Equation Forms

- Mass conservation for this new control volume, after dropping higher order terms, becomes:



$$\oiint_S \rho \bar{V} \cdot \hat{n} dS = 0$$

$$-\rho u A + (\rho + d\rho)(u + du)(A + dA) = 0$$

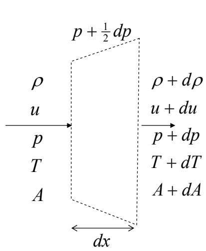
$$\boxed{\rho u dA + \rho A du + u A d\rho = 0}$$

- Or, dividing through by our previous continuity equation, $\rho u A$ = constant:

$$\boxed{\frac{dA}{A} + \frac{du}{u} + \frac{d\rho}{\rho} = 0}$$

Differential Equation Forms [2]

- Similarly for the momentum equation, expanding and dropping higher order terms gives:



$$\oiint_S \rho \bar{V}(\bar{V} \cdot \hat{n}) dS + \oiint_S p \hat{n} dS = 0$$

$$\begin{aligned} &- pA - \rho u^2 A + (p + dp)(A + dA) \\ &\quad + (\rho + d\rho)(u + du)^2 (A + dA) \\ &\quad - \left(p + \tfrac{1}{2}dp\right) dA = 0 \end{aligned}$$

$$\boxed{A dp + \rho u^2 dA + 2\rho u A du + u^2 A d\rho = 0}$$

Differential Equation Forms [3]

- Note that the Euler momentum equation can be obtained by subtracting the first form of continuity times velocity from this momentum equation:

$$A dp + \rho u^2 dA + 2\rho u A du + u^2 A d\rho$$
$$- u(\rho u dA + \rho A du + u A d\rho) = 0$$

$$\boxed{dp + \rho u du = 0}$$

- However, a more interesting result is obtained by using the second form of continuity with the definition of the speed of sound:

$$d\rho = \left(\frac{\partial \rho}{\partial p}\right)_s dp = \frac{dp}{a^2}$$

Differential Equation Forms [4]

- Replacing the density derivative in continuity gives: $\frac{dA}{A} + \frac{du}{u} + \frac{dp}{\rho a^2} = \frac{dA}{A} + \frac{du}{u} - \frac{u du}{a^2} = 0$
- Or, using the definition of Mach number:

$$\boxed{\frac{dA}{A} = (M^2 - 1)\frac{du}{u}}$$

- You might note that this equation, for incompressible flow (M=0) gives the expected inverse relation between velocity and area:

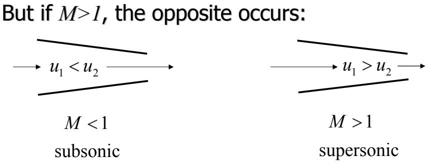
$$\boxed{\frac{dA}{A} = -\frac{du}{u}}$$

Velocity-Area Relations

- The really interesting thing about this equation is what it says about the difference between subsonic and supersonic compressible flow.

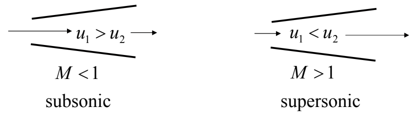
$$\boxed{\frac{dA}{A} = (M^2 - 1)\frac{du}{u}}$$

- If $M<1$, then as area decreases, velocity increases. But if $M>1$, the opposite occurs:

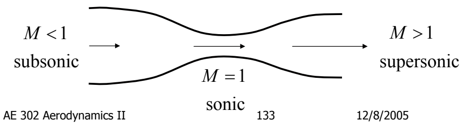


Velocity-Area Relations [2]

- The opposites occur for an increasing area:



- The inflection occurs at M=1 which must be the minimum area through which a flow can pass.
- Thus, the classical shape of a Laval nozzle:



Mach-Area Relation

- The only thing remaining is to try and determine how the flow Mach number varies with area.
- To do this, begin with continuity related to the throat area: $\rho u A = \rho^* u^* A^*$
- This can be re-written as:

$$\frac{A}{A^*} = \frac{\rho^* u^*}{\rho u} = \left(\frac{\rho^*}{\rho_0}\right)\left(\frac{\rho_0}{\rho}\right)\frac{1}{M^*}$$

- But, using previously developed relations and assuming isentropic flow (no shocks yet!):

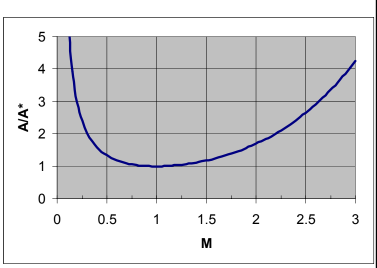
$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma+1}\right)^{1/\gamma-1} \qquad \frac{\rho_0}{\rho} = \left(1+\tfrac{\gamma-1}{2}M^2\right)^{1/\gamma-1} \qquad M^{*2} = \frac{(\gamma+1)M^2}{2+(\gamma-1)M^2}$$

Mach-Area Relation [2]

- Putting all these together gives:

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{(\gamma+1)/(\gamma-1)}$$

- This area ratio is tabulated in the sub- and supersonic sections of the flow tables in your book and NACA 1135.

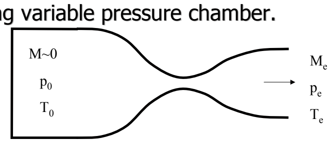


Nozzle Flows

- Up to now, the converging-diverging ducts have all had a sonic throat and supersonic exit.
- However, we now that that is not necessarily the case all the time – otherwise you could generate supersonic flow by squeezing a straw and blowing through it.
- What is missing is the impact of the upstream total pressure and the downstream or exit pressure.
- The combination of these two pressures will determine both if the flow ever goes supersonic – and if it is still supersonic at the exit.

Nozzle Flows [2]

- To see this, consider the special case of a nozzle between a large reservoir of high pressure air and exiting variable pressure chamber.

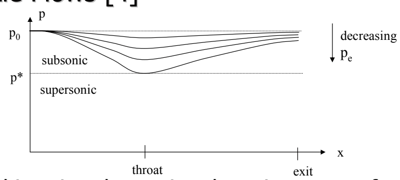


- We'll assume the reservoir is large enough that the velocity is near zero and the pressure remains constant during the time of interest.
- Now, consider what happens for different exit pressures.

Nozzle Flows [3]

- If the exit pressure is equal to the reservoir pressure… nothing happens. There will only be flow if there is a pressure difference.
- If the exit pressure is slightly less then the reservoir pressure, the flow begins to move but only slowly.
- The lower the exit pressure, the higher the velocity up until the velocity at the throat goes sonic.
- We can see this effect by plotting the local pressure versus location in the nozzle.

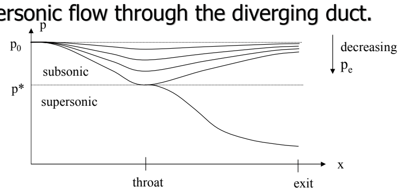
Nozzle Flows [4]



- At this point, decreasing the exit pressure further will not change the converging section any further since it is going as fast as it can at the throat.
- This situation is called having choked flow.
- Note that not even increasing the reservoir pressure would cause an increase in the velocities.

Nozzle Flows [5]

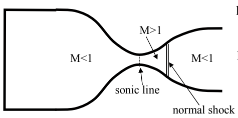
- Now consider the other extreme condition where the exit pressure is low enough to induce supersonic flow through the diverging duct.

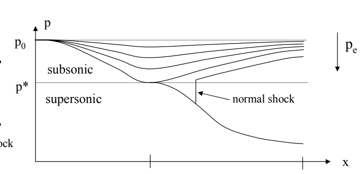


- This flow condition is called fully-expanded.
- But what happens if the exit pressure is some where between these two extremes?

Nozzle Flows [6]

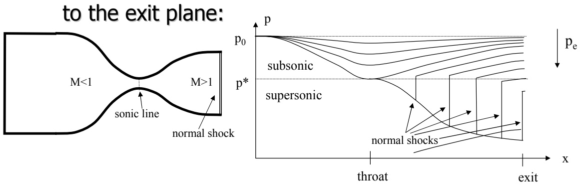
- For pressures below the initial choked condition, the flow goes supersonic beyond the throat.
- However, in order to exit with the correct pressure, the flow then passes through a normal shock, and exits subsonicly.





Nozzle Flows [7]

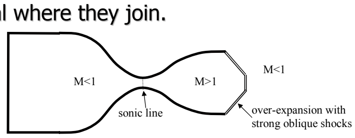
- With decreasing pressure, the shock moves further and further downstream the nozzle – until it gets to the exit plane:



- Note that due to losses in the shock, the exit pressure can be below the upstream p* but the flow is subsonic – i.e, p* decreases across the shock.

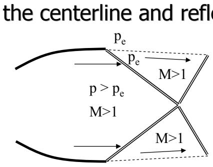
Over Expansion

- If the pressure is dropped further, we get a condition called over expanded flow.
- To get the correct outflow pressure, oblique shocks form on the nozzle lip.
- Initially, these are oblique shocks of the strong type – I.e. subsonic flow behind them – that turn normal where they join.



Over Expansion [2]

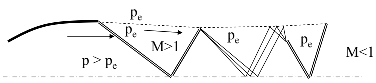
- As the pressure drops further, they become weak oblique waves with supersonic post-shock flow.
- However, since the flow is supersonic, the waves interact at the centerline and reflect as shocks:



- But what happens when the reflected shocks intersect the boundary between the nozzle flow and the ambient air?

Over Expansion [3]

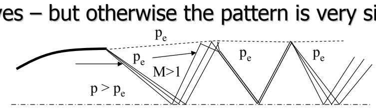
- This is a case we did not discuss earlier – a wave reflecting off of a pressure boundary.
- The rule here is that the wave reflects as a wave of the opposite family and opposite type.
- The change in type is require to ensure equal pressures across the pressure boundary.



- This diamond pattern repeats itself until the flow finally goes subsonic behind one shock.

Under Expansion

- The opposite situation – under expansion – occurs if the exit pressure is below the fully expanded value.
- In this case, the first set of waves are expansion waves – but otherwise the pattern is very similar.

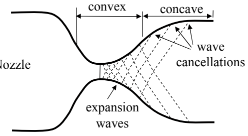


- For optimum performance, there should be no waves – i.e. not over- or under-expanded but fully expanded flow.

Optimal Nozzle Design

- So far, all the analysis performed for supersonic nozzles has relied upon Quasi 1-D assumptions.
- Before we finish, consider the 2-D wave pattern in a real nozzle.

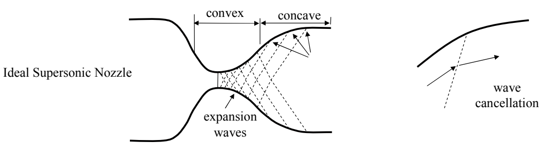
Optimal Supersonic Nozzle



- In the forward nozzle, the convex surface produces expansion waves which accelerate the flow.
- Most nozzles walls turn concave towards the exit to direct the flow into the thrust direction.

Ideal Nozzle Design [2]

- Normally, there would be compression waves in the concave wall region.
- However, in an optimal design, the expansion waves reflected back to the wall can be used to cancel out the compression waves.

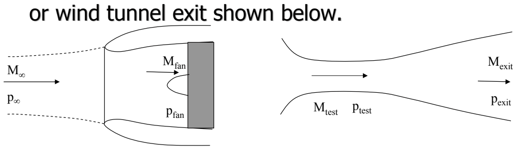


Diffusers

- In the previous section, we concentrated mainly on nozzles: ducts designed to accelerate a flow.
- Let’s also consider diffusers: ducts designed to decelerate a flow.
- Actually, a better description of a diffuser is as a duct designed to achieve a pressure recovery – usually by slowing the flow.
- Since the pressure recovery is important, then the ratio of final to initial total pressure is used as the measure of diffuser efficiency.

Diffusers [2]

- If the flow of interest is subsonic, then diffusers are simply expanding area ducts as in the jet inlet or wind tunnel exit shown below.



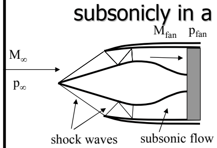
Subsonic Jet Engine Diffuser

Subsonic Wind Tunnel Diffuser

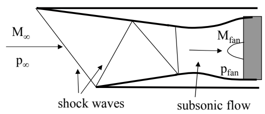
- In both cases, the final velocity is slower and the final pressure higher then initially.
- The limit of how fast the duct area increases is due to the viscous flow on the walls.

Inlet Diffusers

- However, if a flow is initially supersonic, then the flow must first be slowed to Mach 1.0 and then subsonicly in a kind of reverse Laval nozzle.



Conical Diffuser

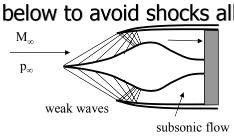


Supersonic 2-D Diffuser

- The most common design is a series of oblique shocks like the two inlet diffusers shown above.
- To make these designs work for a variety of flight speeds, movable surfaces need to be used.

Inlet Diffusers [2]

- Of course, a normal shock can be used like in a ramjet, but with a large loss in efficiency.
- Or you might consider a smooth concave ramp like that below to avoid shocks all together.

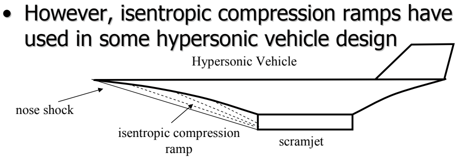


Isentropic Conical Diffuser

- However, this design must be longer and thus have a boundary layer whose total pressure loss is comparable to that of oblique shocks.
- Also this is hard to optimize for different speeds.

Inlet Diffusers [3]

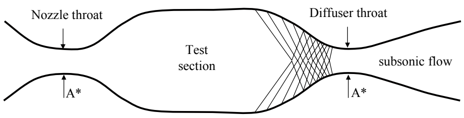
- However, isentropic compression ramps have been used in some hypersonic vehicle design



- At these speeds, the total pressure loss in shock waves is so high so as to make the trade against the boundary layer losses acceptable.

Wind Tunnel Diffusers

- Next consider the diffuser design in a supersonic wind tunnel.
- One possibly design is to use a smooth isentropic like that shown below.

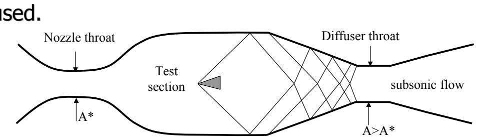


Supersonic Tunnel with Isentropic Diffuser

- In this case there is a downstream throat with the same cross sectional area as the upstream one.

Wind Tunnel Diffusers [2]

- However, with an unknown disturbance in the section, and for a wider range of operating conditions, an oblique shock type diffuser is usually used.

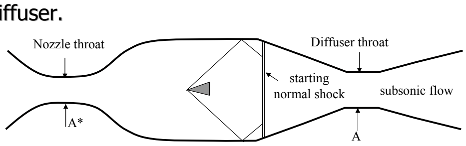


Supersonic Tunnel with Oblique Shock Diffuser

- Also, note that due to losses in the shocks, the downstream throat will have to be slightly larger than the upstream throat.

Wind Tunnel Diffusers [3]

- Another issue with wind tunnel diffuser design it the problem of initially starting the flow.
- It is well known that in the initial start up, after the upstream nozzle goes supersonic, a normal shock passes downstream through the test section to the diffuser.



Supersonic Tunnel with Starting Shock

Wind Tunnel Diffusers [4]

- This normal shock must be “swallowed” into the diffuser throat to complete the start up process.
- However, due to its greater total pressure loss, the throat area required to swallow the normal shock is greater than the area for normal operation.
- Thus, the tunnel must have a variable area diffuser, or compromise between optimum run performance and start-up capability.
- Note that early jet engine designs where notorious for inlet un-starting, I.e. un-swallowing a normal shock due to flow back-up through the engine.