

Compressible Flows

- The bulk of the material presented in the class will deal with the behavior of compressible flows – i.e. those with significant changes in density.
- The cause of density changes in a flow are the pressure changes which also occur when air moves around a body.
- Thus, a measure of compressibility is the fractional change in volume with changes in pressure:

$$\tau = -\frac{dv}{vdp} \equiv \text{compressibility}$$

- In solids, this factor is called the bulk modulus.

Compressible Flows [2]

- To be accurate, the normal differentials in the previous equation should be written as a partials.
- And, if we are dealing with a perfect gas, like air, the gas state only requires knowing 3 variables.
- Thus, the derivative should have a subscript indicating which 3rd variable is being held constant.
- For example, if we were interested in the compressibility of a fluid at constant temperature:

$$\tau_t = -\frac{1}{v} \left(\frac{\partial v}{\partial p} \right)_t \equiv \text{isothermal compressibility}$$

Compressible Flows [3]

- Alternately, it is common to consider the reversible, adiabatic (i.e isentropic) compressibility:

$$\tau_s = -\frac{1}{v} \left(\frac{\partial v}{\partial p} \right)_s \equiv \text{isentropic compressibility}$$

- Both of these factors are properties of the material and can be looked up in references. Typical values are:

$$\tau_t = 5 \times 10^{-10} \frac{\text{m}^2}{\text{N}} \quad \text{for water at 1 atm}$$

$$\tau_t = 1 \times 10^{-5} \frac{\text{m}^2}{\text{N}} \quad \text{for air at 1 atm}$$

- From the magnitude of these we get the commonly observed behavior that water (and all liquids) are much less compressible than air (and all gasses).

Compressible Flows [4]

- However, the compressibility only indicate the general tendency of a fluid to be compressed
- There must also be significant pressure changes which usually means large velocities.
- As a result, the relative flow compressibility is best specified by the Mach number – the ratio of velocity to speed of sound.

$$M = V/a$$

- We will see a little later that the speed of sound is closely related to the compressibility factor, τ .

Compressible Flows [4]

- Thus, the Mach number is essentially the ratio of the likely pressure changes to the compressibility tendency of the fluid.
- We define the usual flight speed regimes by:
 - Incompressible Subsonic: $M < 0.3$
Insignificant density variations
 - Compressible Subsonic: $0.3 < M < 0.7$
Significant density variations, but local flow is always subsonic
 - Transonic: $0.7 < M < 1.2$
Mixture of subsonic and supersonic flow
 - Supersonic: $1.2 < M < 5$
All supersonic flow – temperatures are manageable
 - Hypersonic: $5 < M$
Aero heating is very large and/or real gas effects exist

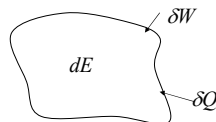
Thermodynamics

- Thermodynamics is the study of the way energy is stored and exchanged – a very important issue in compressible flows.
- A fundamental principle is the **First Law**:
 - The internal energy, E , of a closed system can be increased only by heat added across it's boundaries, δQ , or by work done on the system, δW .

$$dE = \delta Q + \delta W$$

$$\boxed{de = \delta q + \delta w}$$

(per unit mass)



Thermo. - First Law

- Internal energy, E
 - The energy stored in the particles themselves, I.e. random KE, rotation, vibration, chemical bonding, etc.
 - e is specific internal energy, E/m , and de represents a small change in e .
- Work done, δW , and Heat added, δQ
 - The symbol δ represents a small incremental process since both W and Q are methods of exchanging energy and not fluid properties themselves.
 - the work done and heat added per unit mass of fluid are then δw and δq .

Thermo. - Work

- Only one work process is of interest in inviscid aerodynamics: the work done by squeezing a fluid element against the resistance of pressure.
- Consider squeezing a sphere:

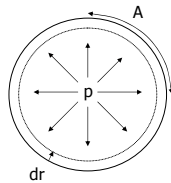
$$\delta w = \frac{F dx}{\text{Mass}} = \frac{p A dr}{\text{Mass}} = - \frac{pdV}{\text{Mass}}$$

– thus

$$\boxed{\delta w = -pdv} = -pd\left(\frac{V}{\rho}\right)$$

- And the first law becomes

$$\delta q = de + pdv$$



Thermo. - Enthalpy

- Before proceeding, we must introduce a new fluid property, Enthalpy, given by the symbol H
- Mathematically, $H = E + PV$
- Or, more commonly, the specific enthalpy is defined by:

$$\boxed{h = e + pv} = e + p / \rho$$

- Enthalpy is the sum of the internal energy and the energy associated with having brought all the particles together into a given volume of space.

Thermo. - Enthalpy [2]

- Small changes in specific internal energy and specific enthalpy are related by:

$$dh = de + d(pv) = de + vdp + pdv$$

- Therefore, the First Law can be rewritten in terms of specific enthalpy:

$$\delta q = dh - vdp = dh - dp/\rho$$

- Note:

- Enthalpy is much more useful in aerodynamics than is internal energy.
- This is because in a flow, total enthalpy (unlike internal energy) is naturally conserved, as we will see a little later.

Thermo. - Specific Heats

- The factors relating changes in temperature to the amount of heat added are called the **Specific Heats**

$$\text{specific heat} = \frac{\delta q}{dT}$$

- Because heat addition is a process, we must also specify the conditions under which heat is added

- At constant volume, $V = \text{constant}$ ($v = \text{constant}$)

$$\delta q = c_v dT \quad c_v = \text{specific heat at constant volume}$$

- From the First Law, however:

$$\delta q = de + pdv^0 \quad \text{thus} \quad de = c_v dT$$

Thermo. - Specific Heats [2]

- At constant pressure, $p = \text{constant}$

$$\delta q = c_p dT \quad c_p = \text{specific heat at constant pressure}$$

- From the First Law, however:

$$\delta q = dh - vdp^0 \quad \text{thus} \quad dh = c_p dT$$

- A little note:

- In many gasses, it is possible to assume that energy is a function of temperature only, i.e. $h(T)$ and $e(T)$.
- This is called being **thermally perfect**.
- For a thermally perfect gas, the above definitions for specific heats are valid **all the time**, whether or not heat is actually added!

Thermo. - Specific Heats [3]

- Further assumptions

- We have already assumed that air is a perfect gas under most conditions, I.e. except for very high ρ 's and low T's.
- Now neglect high T's where we have to worry about vibrational excitation and dissociation of N_2 and O_2 .
- With these restrictions, we can assume that air is **calorically** perfect, or that c_p and c_v are constants.
- The previous relations can then be integrated to get

$$e = c_v T \quad h = c_p T$$

- For convenience, we have set the zero point energies equal to zero, i.e. $e=h=0$ when $T=0$.

Thermo. - Specific Heats [4]

- Note that the specific heats are related, as shown, by the definition of enthalpy:

$$h = e + p v \Rightarrow c_p T = c_v T + RT$$

$$c_p - c_v = R$$

- We will also have many equations which will have the ratio of specific heats as a factor:

$$\gamma = \frac{c_p}{c_v}$$

- This factor is strongly related to the available "modes" of energy storage - I.e. translation, rotation, vibration, electronic.

Thermo. - Specific Heats [5]

- Thus, typical values at room temperature depend upon the molecule type:

$$\gamma = 5/3 = 1.67 \text{ for monatomic gases}$$

$$\gamma = 7/5 = 1.4 \text{ for diatomic gases}$$

$$\gamma \rightarrow 1.1 \text{ for complex, poly-atomic gases}$$

- Also, using this ratio, the specific heats can be written in terms of the specific gas constant by:

$$c_p = \frac{\gamma R}{\gamma - 1} \quad c_v = \frac{R}{\gamma - 1}$$

- Thus, for air with $R = 1716 \text{ ft}^2/\text{sec}^2\text{R}$

$$c_p = 6006 \text{ ft}^2/\text{sec}^2\text{R}$$

$$c_v = 4290 \text{ ft}^2/\text{sec}^2\text{R}$$

Thermo. – Second Law and Entropy

- Another important variable in thermodynamics is called the entropy – often described with the vague idea of the “chaos” of a system.
- In gasses, this chaos is the number of different ways the total energy of the system can be distributed among the available energy states.
- A good analogy is to consider how many different combinations of coins and bills you might have if you had \$9.50 in your pocket.
- Obviously, the more energy (or money), the more combinations there might be – and the higher the chaos.

Thermo. – Second Law and Entropy [2]

- The determination of absolute entropy of a system can be very complex due to all the energy modes.
- However, the change of entropy is known precisely from the 2nd Law of thermodynamics:

$$ds = \frac{\delta q}{T}$$

- Effectively, this equation is the ratio of the energy added to a system to that which is already present.
- The equation above represents a perfect world. In practice, the chaos of a system also changes whenever a system undergoes a non-equilibrium process.

Thermo. – Second Law and Entropy [2]

- Examples of non-equilibrium process are friction, heat flux, or diffusion – or simply very fast changes.
- Non-equilibrium process are **irreversible** – they only proceed naturally in one direction.
- As a result, the full 2nd Law may be written as:

$$ds = \frac{\delta q}{T} + ds_{\text{irrev}} \quad \text{or} \quad ds \geq \frac{\delta q}{T}$$

- Unfortunately, these equations are much use for practical calculations by themselves.
- To be of use, the definition of entropy must be combined with the 1st law.

Thermo. – Entropy Calculation

- The 1st law can be written in terms of either internal energy or enthalpy as:

$$\delta q = de + pdv \qquad \delta q = dh - vdp$$

- Using the definition of entropy given before, these become:

$$Tds = c_v dT + pdv \qquad Tds = c_p dT - vdp$$

$$ds = c_v \frac{dT}{T} + p \frac{dv}{T} \qquad ds = c_p \frac{dT}{T} - v \frac{dp}{T}$$

$$ds = c_v \frac{dT}{T} + R \frac{dv}{v} \qquad ds = c_p \frac{dT}{T} - R \frac{dp}{p}$$

Thermo. – Entropy Calculation [2]

- These equations can be integrated between initial and final conditions for a calorically perfect gas:

$$s_2 - s_1 = c_v \ln\left(\frac{T_2}{T_1}\right) + R \ln\left(\frac{v_2}{v_1}\right) = c_v \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{\rho_2}{\rho_1}\right)$$

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{p_2}{p_1}\right)$$

- These relations will later come in very handy in calculating the entropy change across shocks.
- You may also recall using these relations in ES305 to predict the entropy at different points in a cycle.

Isentropic Flow

- Now consider a special situation.
- First, assume no heat flux, $\delta q = 0$ or **adiabatic**
 - A reasonable assumption; we don't often try to heat or cool the air around an airplane!
 - This **does not** mean the temperature remains constant - doing work can still change the energy of the fluid and thus it's temperature.
- Also, let's assume the flow is **reversible**
 - Thus, no friction - a reasonable assumption everywhere but near the skin surface.
 - And also that there are not abrupt property changes. Abrupt changes induce dissipative losses.

Thermo. - Isentropic flow

- A flow which is adiabatic **and** reversible has constant entropy and is called **isentropic**.

– Practically, this means that some special relations exist between our fluid properties.

– To see this, start with the reversible, adiabatic energy equations:

$$\delta q^{*0} = de + pdv \quad \delta q^{*0} = dh - vdp$$

– including the definitions of the specific heats:

$$0 = c_v dT + pdv \quad 0 = c_p dT - vdp$$

– rearrange and divide:

$$\frac{c_p dT}{c_v dT} = \gamma = \frac{vdp}{-pdv} \quad \text{or} \quad \frac{dp}{p} = -\gamma \frac{dv}{v}$$

Thermo. - Isentropic flow [2]

– Now, integrate over the change from one condition, 1, to another, 2:

$$\ln\left(\frac{p_2}{p_1}\right) = -\gamma \ln\left(\frac{v_2}{v_1}\right) \quad \text{or} \quad \left(\frac{p_2}{p_1}\right) = \left(\frac{v_2}{v_1}\right)^{-\gamma} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$$

– We can also use the perfect gas law to introduce T

$$\left(\frac{p_2}{p_1}\right) = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{p_2}{RT_2} \frac{RT_1}{p_1}\right)^{\gamma} \quad \text{or} \quad \left(\frac{p_2}{p_1}\right) = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$$

Thermo. - Isentropic flow [3]

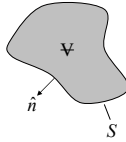
– To summarize, for isentropic flow, r , p and T are related by

$$\left(\frac{p_2}{p_1}\right) = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$$

- The above equations effectively simplifies having to know 3 variables to only 2 to define a state.
- This make sense since for isentropic flow, one state variable, s , is constant.
- These equations can be used for many flow except for boundary layers and across shockwaves.

Flow Conservation Laws

- Let's now derive the basic conservation laws for fluid flow.
- We'll use a control volume approach which implies:
 - We have a volume in space of fixed size
 - Fluid passes freely through the surface
- For this case, a general statement of conservation is:



$$\left[\begin{array}{l} \text{The time rate of change of} \\ \text{a property in the volume} \end{array} \right] = \left[\begin{array}{l} \text{The net rate at which} \\ \text{the property flow in} \end{array} \right] + \left[\begin{array}{l} \text{The rate of production} \\ \text{of that property inside} \end{array} \right]$$

Flow Conservation Laws [2]

- Apply this general rule to our three conserved properties: mass, momentum and energy.
- For mass, which cannot be produced, the equation becomes:

$$\frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho d\mathcal{V} = - \iint_S \rho \vec{V} \cdot \hat{n} dS$$
- The right hand integral represents the net flux through the boundaries ($\rho V A$) and is negative for a net influx – thus the negative sign.
- For momentum, we must consider each axial direction separately.
- Also, external forces, like friction and pressure, can “produce” momentum inside the control volume.

Flow Conservation Laws [3]

- Considering only pressure forces, for x momentum conservation, the equation is:

$$\frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho u d\mathcal{V} = - \iint_S \rho u \vec{V} \cdot \hat{n} dS - \iint_S p \hat{i} \cdot \hat{n} dS$$
- The dot product in the pressure term gives the x component of the force, but is negative for a momentum producing force... thus the negative sign.
- Similarly, for y momentum we would have:

$$\frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho v d\mathcal{V} = - \iint_S \rho v \vec{V} \cdot \hat{n} dS - \iint_S p \hat{j} \cdot \hat{n} dS$$
- A vector momentum conservation equation can be formed by adding these two, multiplied by their unit vectors to get:

$$\frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho \vec{V} d\mathcal{V} = - \iint_S \rho \vec{V} (\vec{V} \cdot \hat{n}) dS - \iint_S p \hat{n} dS$$

Flow Conservation Laws [4]

- Next, for energy conservation the specific energy is composed of two forms: internal and kinetic.
- Note that we usually neglect gravity (potential energy) for air flow.
- Also, pressure forces do work on the control volume by pushing with or against the flow direction – this is the so-called Flow Work.
- The equation for energy conservation is then:

$$\frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho \left(e + \frac{1}{2} V^2 \right) d\mathcal{V} = - \iint_S \rho \left(e + \frac{1}{2} V^2 \right) \vec{V} \cdot \hat{n} dS - \iint_S p (\vec{V} \cdot \hat{n}) dS$$
- This again neglects friction and also heat addition.

Flow Conservation Laws [5]

- To summarize, we have the following inviscid, adiabatic flow conservation equations:

$$\frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho d\mathcal{V} = - \iint_S \rho \vec{V} \cdot \hat{n} dS$$

$$\frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho \vec{V} d\mathcal{V} = - \iint_S \rho \vec{V} (\vec{V} \cdot \hat{n}) dS - \iint_S p \hat{n} dS$$

$$\frac{\partial}{\partial t} \iiint_{\mathcal{V}} \rho \left(e + \frac{1}{2} V^2 \right) d\mathcal{V} = - \iint_S \rho \left(e + \frac{1}{2} V^2 \right) \vec{V} \cdot \hat{n} dS - \iint_S p (\vec{V} \cdot \hat{n}) dS$$

- Jointly, these equations are often called the unsteady Euler equations– although they can be written in many other ways.
- The equations with viscous and heat flux terms, which we will see later, are called the Navier-Stokes equations.
